

1. 求微分方程 $y'' = \frac{1}{\sqrt{1-x^2}}$ 的通解.

$$\text{解 } y' = \int y'' dx = \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C_1$$

$$y = \int y' dx = \int (\arcsin x + C_1) dx = \int \arcsin x dx + C_1 x$$

$$= x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} dx + C_1 x$$

$$= x \arcsin x + \sqrt{1-x^2} + C_1 x + C_2$$

2. 求微分方程 $y'' = y' + x$ 的通解

解: 设 $y' = p$, 则 $y'' = p'$, 于是原方程化为

$$p' - p = x$$

$$p = e^{-\int -1 dx} \left(\int x e^{\int -1 dx} dx + C_1 \right)$$

$$= e^x (\int x e^{-x} dx + C_1)$$

$$= e^x (\int -x de^{-x} + C_1) = e^x (-x e^{-x} - e^{-x} + C_1)$$

$$= -x - 1 + C_1 e^x$$

$$y = \int p dx = \int (-x - 1 + C_1 e^x) dx$$

$$= -\frac{1}{2} x^2 - x + C_1 e^x + C_2$$

3. 求微分方程 $y'' = (y')^3 + y'$ 的通解

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解 设 $u = y'$, 则 $u' = y''$, 原方程化为

$$\frac{du}{dx} = u^3 + u$$

分离变量 $\frac{du}{u^3 + u} = dx$

两边积分 $\int \frac{du}{u^3 + u} = \int dx = x + \tilde{C}_1$

即 $\ln \frac{u}{\sqrt{u^2 + 1}} = x + \tilde{C}_1$ $\frac{u^2}{u^2 + 1} = e^{2x + 2\tilde{C}_1} = \tilde{C}_2 e^{2x}$

$$u = \sqrt{\frac{\tilde{C}_2 e^{2x}}{1 - \tilde{C}_2 e^{2x}}}$$

$$y = \int y' dx = \int \sqrt{\frac{\tilde{C}_2 e^{2x}}{1 - \tilde{C}_2 e^{2x}}} dx$$

设 $\sqrt{\tilde{C}_2} e^x = \sin \theta$ $x = \ln \frac{\sin \theta}{\sqrt{\tilde{C}_2}}$

$$dx = \frac{\sqrt{\tilde{C}_2}}{\sin \theta} \cos \theta / \sqrt{\tilde{C}_2} d\theta$$

$$= \cot \theta d\theta$$

于是, 原式 $y = \int \sqrt{\frac{\tilde{C}_2 e^{2x}}{1 - \tilde{C}_2 e^{2x}}} dx = \int \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} d\theta = \theta + \tilde{C}_3$

$$= \arcsin \sqrt{\tilde{C}_2} e^x + \tilde{C}_3$$

$$= \arcsin C_1 e^x + C_2$$

这里 $\tilde{C}_2 = e^{2\tilde{C}_1}$ $C_1 = \sqrt{\tilde{C}_2}$ $C_2 = \tilde{C}_3$

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4. 试求 $y''=x$ 的经过点 $M(0,1)$ 且在此点与直线 $y=\frac{x}{2}+1$ 相切的积分曲线

解: $y' = \int y'' dx = \int x dx = \frac{1}{2}x^2 + C_1$

直线 $y=\frac{x}{2}+1$ 在 $M(0,1)$ 点的斜率 $k|_{(0,1)} = \frac{1}{2}$

于是 $\frac{1}{2} \cdot 0^2 + C_1 = \frac{1}{2} \Rightarrow C_1 = \frac{1}{2}$

所以 $y' = \frac{1}{2}x^2 + \frac{1}{2}$

$$y = \int y' dx = \int \left(\frac{1}{2}x^2 + \frac{1}{2}\right) dx = \frac{1}{6}x^3 + \frac{1}{2}x + C_2$$

$$y(0)=1 \Rightarrow 1 = \frac{1}{6} \cdot 0^3 + \frac{1}{2} \cdot 0 + C_2 \Rightarrow C_2 = \cancel{0} 1$$

$$\therefore y = \frac{1}{6}x^3 + \frac{1}{2}x + \cancel{0} 1$$

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5. 求微分方程 $y'' = e^{2y}$ 满足初始条件 $y|_{x=0} = y'|_{x=0} = 0$ 的特解.

解: 设 $y' = p$, $\frac{dp}{dy} = \frac{dp}{dx} \cdot \frac{dx}{dy} = y'' \cdot \frac{1}{p}$, $y'' = p \frac{dp}{dy}$

微分方程化为 $p \cdot \frac{dp}{dy} = e^{2y}$

分离变量 $p dp = e^{2y} dy$

两边积分 $\int p dp = \int e^{2y} dy$

解得 $\frac{1}{2} p^2 = \frac{1}{2} e^{2y} + C$

由 $y|_{x=0} = y'|_{x=0} = 0$ 得 $\frac{1}{2} \cdot 0^2 = \frac{1}{2} \cdot e^{2 \cdot 0} + C \Rightarrow C = -\frac{1}{2}$

所以 $p = \sqrt{e^{2y} - 1}$

$$\frac{dy}{dx} = \sqrt{e^{2y} - 1}$$

分离变量 $\frac{dy}{\sqrt{e^{2y} - 1}} = dx$

两边积分 $\int \frac{dy}{\sqrt{e^{2y} - 1}} = \int dx$

设 $e^y = \sec \theta = \frac{1}{\cos \theta}$ $y = \ln \frac{1}{\cos \theta} = -\ln \cos \theta$ $dy = -\frac{1}{\cos \theta} \cdot (-\sin \theta) = \frac{\sin \theta}{\cos \theta} d\theta = \tan \theta d\theta$

$$\int \frac{dy}{\sqrt{e^{2y} - 1}} = \int \frac{1}{\tan \theta} \cdot \tan \theta d\theta = \theta + C_1 = \arccos e^{-y} + C_1$$

$$\therefore \arccos e^{-y} + C_1 = x \quad \text{由 } y|_{x=0} = 0 \Rightarrow C_1 = 0$$

$$\therefore \arccos e^{-y} = x \quad \text{即 } e^{-y} = \cos x$$

也可以化为 $e^y = \sec x$ (或 $y = \ln \sec x$)